Periodic oscillatory solution of bidirectional associative memory networks with delays

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Periodic oscillatory phenomena of bidirectional associative memory (BAM) networks with axonal signal transmission delays are investigated by constructing suitable Lyapunov functionals and some analysis techniques. Some simple sufficient conditions are derived ensuring the existence and uniqueness of periodic oscillatory solutions of the BAM with delays, and all other solutions of the BAM converge exponentially to a periodic oscillatory solution. These conditions are presented in terms of system parameters, and have an important leading significance in the design and applications of periodic oscillatory neural circuits for the BAM with delays.

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I. INTRODUCTION

Recently, a class of two-layer heteroassociative networks called bidirectional associative memory (BAM) networks with and without axonal signal transmission delays was proposed in Refs. [1–5], and has been used to obtain important advances in many fields such as pattern recognition and automatic control. It is well known that the collective computational capabilities of networks (such as optimization, associative memory, and oscillation) depend on the dynamic behavior of certain neural networks. There has been a steady increase in interest in the potential applications of the dynamics of artificial neural networks in signal and image processing. The stability of bidirectional associative memory neural networks without and with delays has been studied (we refer to Refs. [1–9] and the references cited therein). Moreover, studies on neural dynamical systems not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior, bifurcation, and chaos. In many applications, the properties of periodic oscillatory solutions are of great interest. For example, the human brain is in periodic oscillatory or chaos states, hence it is of prime importance to study periodic oscillatory and chaos phenomena of neural networks. To the best of our knowledge, few authors have considered periodic oscillatory solutions for BAM networks with delays. In this paper, we shall give some simple sufficient conditions for the existence and uniqueness of periodic oscillatory solutions of BAM networks with delays by constructing a suitable Lyapunov functional and some analysis techniques; all other solutions of the BAM converge exponentially to the periodic oscillatory solution. These possess an important leading significance in the design and applications of periodic oscillatory BAM with delays, and are of great interest in many applications.

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II. PERIODIC OSCILLATORY SOLUTIONS OF THE BAM WITH DELAYS

In this paper, we consider the periodic oscillatory solutions of BAM networks with delays described by differential equations with delays:

$$x_i'(t)=-a_ix_i(t)+\sum_{j=1}^{p} w_{ji}s_j(y_j(t-\tau_{ji}))+I_i(t),i=1,2,\ldots,n \tag{1}$$

$$y_j'(t)=-b_jy_j(t)+\sum_{i=1}^{n} v_{ij}s_i(x_i(t-\sigma_{ij}))+I_j(t),\quad j=1,2,\ldots,p \tag{2}$$

in which $x=[x_1,x_2,\ldots,x_n]^T\in R^n, y=[y_1,y_2,\ldots,y_p]^T\in R^p$, and a_i,b_j,τ_{ij}, and σ_{ij} are nonnegative constants. The signal functions s_i possess the following properties:

Hypothesis 1. s_i is bounded on R;

Hypothesis 2. There is a number $\mu_i>0$ such that $|s_i(u)-s_i(v)|\leq\mu_i|u-v|$ for any $u,v\in R$.

Time delays τ_{ji} and σ_{ij} correspond to a finite speed of the axonal signal transmission; $\tau=\max_{1\leq i\leq n,1\leq j\leq p}(\tau_{ji}),\sigma=\max_{1\leq i\leq n,1\leq j\leq p}(\sigma_{ij})$; and the external inputs $I_i:R^+\to R$, $i=1,2,\ldots,n$ and $J_j:R^+\to R$, $j=1,2,\ldots,p$ are continuously periodic functions with period ω, i.e., $I_i(t+\omega)=I_i(t),J_j(t+\omega)=J_j(t)$. An analog circuit implementing of BAM equations (1) and (2) with axonal signal transmission delays can be seen in Refs. [5,9] [here $a_i=1/(R_iC_i),b_j=1/(R_jC_j),I_i=I_i(t)$, and $J_j=J_j(t)$, $i=1,2,\ldots,n$ and $j=1,2,\ldots,p$].

Theorem 1. For the BAM equations (1) and (2), suppose that the output functions s_i satisfy Hypotheses (1) and (2) above, and that there exist constants $\lambda_i>0,\lambda_{n+j}>0$ ($i=1,2,\ldots,n$ and $j=1,2,\ldots,p$) such that

$$-\lambda_ia_i+\sum_{j=1}^{p}\lambda_{n+j}\mu_i|v_{ij}|<0,\quad i=1,2,\ldots,n$$

and

$$-\lambda_{n+j}b_j+\sum_{i=1}^{n}\lambda_i\mu_j|w_{ji}|<0,\quad j=1,2,\ldots,p,$$

in which μ_j are constant numbers of Hypothesis (2) above. Then there exists exactly one ω-periodic solution of BAM equations (1) and (2), and all other solutions of BAM equations (1) and (2) converge exponentially to it as $t\to+\infty$.

Proof. Let $C=C\left(\binom{[-\tau,0]}{[-\sigma,0]},\ R^{n+p}\right)$ be the Banach space of continuous functions which map $\binom{[-\tau,0]}{[-\sigma,0]}$ into R^{n+p} with the topology of uniform convergence. For any $\binom{\varphi_x}{\varphi_y}\in C$, we define

$$\left\|\begin{pmatrix}\varphi_x\\ \varphi_y\end{pmatrix}\right\|=\sup_{-\tau\leqslant\theta\leqslant 0}|\varphi_x(\theta)|+\sup_{-\sigma\leqslant\theta\leqslant 0}|\varphi_y(\theta)|,$$

in which

$$|\varphi_x(\theta)|=\sum_{i=1}^{n}|\varphi_{xi}(\theta)|,|\varphi_y(\theta)|=\sum_{i=1}^{p}|\varphi_{yi}(\theta)|.$$

For all $\binom{\varphi_x}{\varphi_y},\binom{\psi_x}{\psi_y}\in C$, we denote the solutions of BAM equations (1) and (2) through $\left(\binom{0}{0},\binom{\varphi_x}{\varphi_y}\right)$ and $\left(\binom{0}{0},\binom{\psi_x}{\psi_y}\right)$ as

$$x(t,\varphi_x)=(x_1(t,\varphi_x),x_2(t,\varphi_x),\ldots,x_n(t,\varphi_x))^T,y(t,\varphi_y)=(y_1(t,\varphi_y),y_2(t,\varphi_y),\ldots,y_p(t,\varphi_y))^T,$$

$$x(t,\psi_x)=(x_1(t,\psi_x),x_2(t,\psi_x),\ldots,x_n(t,\psi_x))^T,\quad\text{and}$$

$$y(t,\psi_y)=(y_1(t,\psi_y),y_2(t,\psi_y),\ldots,y_p(t,\psi_y))^T,$$

respectively.

Define

$$x_t(\varphi_x)=x(t+\theta,\varphi_x),\quad \theta\in[-\tau,0],\quad t\geqslant 0,$$

$$y_t(\varphi_y)=y(t+\theta,\varphi_y),\quad \theta\in[-\sigma,0],\quad t\geqslant 0;$$

then $\binom{x_t(\varphi_x)}{y_t(\varphi_y)}\in C\ \ \forall\ \ t\geqslant 0$.

Thus it follows from BAM equations (1) and (2) that

$$[x_i(t,\varphi_x)-x_i(t,\psi_x)]'=-a_i(x_i(t,\varphi_x)-x_i(t,\psi_x))+\sum_{j=1}^{p}w_{ji}[s_j(y_j(t-\tau_{ji},\varphi_y))-s_j(y_j(t-\tau_{ji},\psi_y))],\tag{3}$$

$$[y_j(t,\varphi_y)-y_j(t,\psi_y)]'=-b_j(y_j(t,\varphi_y)-y_j(t,\psi_y))+\sum_{i=1}^{n}v_{ij}[s_i(x_i(t-\sigma_{ij},\varphi_x))-s_i(x_i(t-\sigma_{ij},\psi_x))]\tag{4}$$

for $t\geqslant 0$, where $i=1,2,\ldots,n$ and $j=1,2,\ldots,p$. Since

$$-\lambda_i a_i+\sum_{j=1}^{p}\lambda_{n+j}\mu_i|v_{ij}|<0,\quad i=1,2,\ldots,n$$

and

$$-\lambda_{n+j}b_j+\sum_{i=1}^{n}\lambda_i\mu_j|w_{ji}|<0,\quad j=1,2,\ldots,p,$$

we can choose a small $\varepsilon>0$ such that

$$\lambda_i(\varepsilon-a_i)+e^{\varepsilon\sigma}\sum_{j=1}^{p}\lambda_{n+j}\mu_i|v_{ij}|<0,\quad i=1,2,\ldots,n$$

and

$$\lambda_{n+j}(\varepsilon-b_j)+e^{\varepsilon\tau}\sum_{i=1}^{n}\lambda_i\mu_j|w_{ji}|<0,\quad j=1,2,\ldots,p.$$

We consider the Lyapunov functional

$$V(t)=\sum_{i=1}^{n}\lambda_i\Bigg[|x_i(t,\varphi_x)-x_i(t,\psi_x)|e^{\varepsilon t}+\sum_{j=1}^{p}\int_{t-\tau_{ji}}^{t}|w_{ji}||s_j(y_j(s,\varphi_y))-s_j(y_j(s,\psi_y))|e^{\varepsilon(s+\tau_{ji})}ds\Bigg]+\sum_{j=1}^{p}\lambda_{n+j}\Bigg[|y_j(t,\varphi_y)-y_j(t,\psi_y)|e^{\varepsilon t}+\sum_{i=1}^{n}\int_{t-\sigma_{ij}}^{t}|v_{ij}||s_i(x_i(s,\varphi_x))-s_i(x_i(s,\psi_x))|e^{\varepsilon(s+\sigma_{ij})}ds\Bigg].$$

Calculating the upper right Dini derivative D^+V of V along the solution of Eqs. (3) and (4), we have

$$
\begin{aligned}
D^+V(t)|_{(3)-(4)} = & \sum_{i=1}^{n} \lambda_i \Bigg[D^+(|x_i(t,\varphi_x)-x_i(t,\psi_x)|e^{\varepsilon t})|_{(3)} + \sum_{j=1}^{p} |w_{ji}||s_j(y_j(t,\varphi_y))-s_j(y_j(t,\psi_y))|e^{\varepsilon(t+\tau_{ji})} \\
& - \sum_{j=1}^{p} |w_{ji}||s_j(y_j(t-\tau_{ji},\varphi_y))-s_j(y_j(t-\tau_{ji},\psi_y))|e^{\varepsilon t} \Bigg] + \sum_{j=1}^{p} \lambda_{n+j} \Bigg[D^+(|y_j(t,\varphi_y)-y_j(t,\psi_y)|e^{\varepsilon t})|_{(4)} \\
& + \sum_{i=1}^{n} |v_{ij}||s_i(x_i(t,\varphi_x))-s_i(x_i(t,\psi_x))|e^{\varepsilon(t+\sigma_{ij})} - \sum_{i=1}^{n} |v_{ij}||s_i(x_i(t-\sigma_{ij},\varphi_x))-s_i(x_i(t-\sigma_{ij},\psi_x))|e^{\varepsilon t} \Bigg] \\
\leqslant & \sum_{i=1}^{n} \lambda_i \Bigg[(\varepsilon - a_i)|x_i(t,\varphi_x)-x_i(t,\psi_x)|e^{\varepsilon t} + e^{\varepsilon t}e^{\varepsilon\tau} \sum_{j=1}^{p} |w_{ji}||s_j(y_j(t,\varphi_y))-s_j(y_j(t,\psi_y))| \Bigg] \\
& + \sum_{j=1}^{p} \lambda_{n+j} \Bigg[(\varepsilon - b_j)|y_j(t,\varphi_y)-y_j(t,\psi_y)|e^{\varepsilon t} + e^{\varepsilon t}e^{\varepsilon\sigma} \sum_{i=1}^{n} |v_{ij}||s_i(x_i(t,\varphi_x))-s_i(x_i(t,\psi_x))| \Bigg] \\
\leqslant & e^{\varepsilon t} \sum_{i=1}^{n} \Bigg[\lambda_i(\varepsilon - a_i) + e^{\varepsilon\sigma} \sum_{j=1}^{p} \lambda_{n+j}\mu_i|v_{ij}| \Bigg] |x_i(t,\varphi_x)-x_i(t,\psi_x)| + e^{\varepsilon t} \sum_{j=1}^{p} \Bigg[\lambda_{n+j}(\varepsilon - b_j) \\
& + e^{\varepsilon\tau} \sum_{i=1}^{n} \lambda_i \mu_j |w_{ji}| \Bigg] |y_j(t,\varphi_y)-y_j(t,\psi_y)| \\
\leqslant & 0,
\end{aligned}
$$

and so

$$
V(t) \leqslant V(0), \quad t \geqslant 0,
$$

since

$$
e^{\varepsilon t}(\min_{1\leqslant i\leqslant n+p} \lambda_i) \left(\sum_{i=1}^{n} |x_i(t,\varphi_x)-x_i(t,\psi_x)| + \sum_{j=1}^{p} |y_j(t,\varphi_y)-y_j(t,\psi_y)| \right) \leqslant V(t), \quad t \geqslant 0,
$$

$$
\begin{aligned}
V(0) = & \sum_{i=1}^{n} \lambda_i \Bigg[|\varphi_{xi} - \psi_{xi}| + \sum_{j=1}^{p} \int_{-\tau_{ji}}^{0} |w_{ji}||s_j(y_j(s,\varphi_y))-s_j(y_j(s,\psi_y))|e^{\varepsilon(s+\tau_{ji})} ds \Bigg] \\
& + \sum_{j=1}^{p} \lambda_{n+j} \Bigg[|\varphi_{yj} - \psi_{yj}| + \sum_{i=1}^{n} \int_{-\sigma_{ij}}^{0} |v_{ij}||s_i(x_i(s,\varphi_x))-s_i(x_i(s,\psi_x))|e^{\varepsilon(s+\sigma_{ij})} ds \Bigg] \\
\leqslant & \Bigg[\max_{1\leqslant i\leqslant n} \lambda_i + \max_{1\leqslant i\leqslant n} (\mu_i) e^{\varepsilon\sigma} \sigma \sum_{j=1}^{p} \lambda_{n+j} \max_{1\leqslant i\leqslant n} |v_{ij}| \Bigg] \|\varphi_x - \psi_x\| + \Bigg[\max_{1\leqslant j\leqslant p} \lambda_{n+p} + \max_{1\leqslant j\leqslant p} (\mu_j) e^{\varepsilon\tau} \tau \sum_{i=1}^{n} \lambda_i \max_{1\leqslant j\leqslant p} |w_{ji}| \Bigg] \|\varphi_y - \psi_y\|.
\end{aligned}
$$

Then it easily follows that

$$
\sum_{i=1}^{n} |x_i(t,\varphi_x)-x_i(t,\psi_x)| + \sum_{j=1}^{p} |y_j(t,\varphi_y)-y_j(t,\psi_y)| \leqslant k(\|\varphi_x - \psi_x\| + \|\varphi_y - \psi_y\|)e^{-\varepsilon t}
$$

for all $t \geqslant 0$, where $k \geqslant 1$ is a constant. Hence we have

$$
\sum_{i=1}^{n} |x_i(t,\varphi_x)-x_i(t,\psi_x)| \leqslant k(\|\varphi_x - \psi_x\| + \|\varphi_y - \psi_y\|)e^{-\varepsilon t},
$$

$$
\sum_{j=1}^{p} |y_j(t,\varphi_y)-y_j(t,\psi_y)| \leqslant k(\|\varphi_x - \psi_x\| + \|\varphi_y - \psi_y\|)e^{-\varepsilon t}.
$$

Thus

$$
\|x_t(\varphi_x) - x_t(\psi_x)\| \leqslant k e^{-\varepsilon(t-\tau)}(\|\varphi_x - \psi_x\| + \|\varphi_y - \psi_y\|),
$$

$$
\|y_t(\varphi_y) - y_t(\psi_y)\| \leqslant k e^{-\varepsilon(t-\sigma)}(\|\varphi_x - \psi_x\| + \|\varphi_y - \psi_y\|)
$$

for $t \geqslant 0$. We can choose a positive integer m such that

$$
k e^{-\varepsilon(m\omega - \tau)} \leqslant \frac{1}{4}, \quad k e^{-\varepsilon(m\omega - \sigma)} \leqslant \frac{1}{4}.
$$

Now define a Poincare mapping $P: C \to C$ by

$$
P\begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix} = \begin{pmatrix} x_\omega(\varphi_x) \\ y_\omega(\varphi_y) \end{pmatrix}.
$$

Then we can derive from BAM equations (1) and (2) that

$$\left\|P^m\begin{pmatrix}\varphi_x\\ \varphi_y\end{pmatrix}-P^m\begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix}\right\|\leqslant\frac{1}{2}\left\|\begin{pmatrix}\varphi_x\\ \varphi_y\end{pmatrix}-\begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix}\right\|.$$

This implies that P^m is a contraction mapping; hence there exists a unique fixed point $\binom{\varphi_x^*}{\varphi_y^*}\in C$ such that $P^m\binom{\varphi_x^*}{\varphi_y^*}=\binom{\varphi_x^*}{\varphi_y^*}$.

Note that

$$P^m\left(P\begin{pmatrix}\varphi_x^*\\ \varphi_y^*\end{pmatrix}\right)=P\left(P^m\begin{pmatrix}\varphi_x^*\\ \varphi_y^*\end{pmatrix}\right)=P\begin{pmatrix}\varphi_x^*\\ \varphi_y^*\end{pmatrix}.$$

This shows that $P\binom{\varphi_x^*}{\varphi_y^*}\in C$ is also a fixed point of P^m, and so $P\binom{\varphi_x^*}{\varphi_y^*}=\binom{\varphi_x^*}{\varphi_y^*}$, i.e.,

$$\begin{pmatrix}x_\omega(\varphi_x^*)\\ y_\omega(\varphi_y^*)\end{pmatrix}=\begin{pmatrix}\varphi_x^*\\ \varphi_y^*\end{pmatrix}.$$

Let $\binom{x(t,\varphi_x^*)}{y(t,\varphi_y^*)}$ be the solution of Eqs. (1) and (2) through $(\binom{0}{0},\binom{\varphi_x^*}{\varphi_y^*})$; then $\binom{x(t+\omega,\varphi_x^*)}{y(t+\omega,\varphi_y^*)}$ is also a solution of BAM equations (1) and (2). Obviously,

$$\begin{pmatrix}x_{t+\omega}(\varphi_x^*)\\ y_{t+\omega}(\varphi_y^*)\end{pmatrix}=\begin{pmatrix}x_t(x_\omega(\varphi_x^*))\\ y_t(y_\omega(\varphi_y^*))\end{pmatrix}=\begin{pmatrix}x_t(\varphi_x^*)\\ y_t(\varphi_y^*)\end{pmatrix}$$

for $t\geqslant 0$; hence

$$\begin{pmatrix}x(t+\omega,\varphi_x^*)\\ y(t+\omega,\varphi_y^*)\end{pmatrix}=\begin{pmatrix}x(t,\varphi_x^*)\\ y(t,\varphi_y^*)\end{pmatrix}.$$

This shows that $\binom{x(t,\varphi_x^*)}{y(t,\varphi_y^*)}$ is exactly one ω-periodic solution of BAM equations (1) and (2) and it is easy to see that all other solutions of BAM equations (1) and (2) converge exponentially to it as $t\to+\infty$.

Applying theorem 1 above, we easily prove the following theorem.

Theorem 2. For the BAM equations (1) and (2), suppose that the outputs functions s_i satisfy Hypotheses (1) and (2) above, and

$$-a_i+\sum_{j=1}^{p}\mu_i|v_{ij}|<0,\quad -b_j+\sum_{i=1}^{n}\mu_j|w_{ji}|<0,$$

$$i=1,2,\ldots,n,\quad j=1,2,\ldots,p$$

in which μ_j is constant numbers of Hypotheses (2) above. Then there exists exactly one ω-periodic solution of BAM equations (1) and (2), and all other solutions of BAM equations (1) and (2) converge exponentially to it as $t\to+\infty$.

III. CONCLUSIONS

In this paper, we have presented some simple sufficient conditions in term of systems parameters for periodic oscillatory solutions of BAM equations (1) and (2) with delays, which all other solutions of the BAM converge exponentially to as $t\to+\infty$. The conditions possess highly important significance in some applied fields; for instance, they can be applied to design a globally, exponentially periodic, oscillatory BAM, and easily checked in practice by simple algebraic methods. These play an important role in the design and applications of BAM. In addition, the methods of this paper may be extended to study some other systems such as cellular neural networks with delays and Hopfield neural networks, and so on; see Refs. [10–20]. The discussion presented here assumes a constant signal transmission delay. More general results related to the case of time-varying delay will be reported later.

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